

Machine Learning Worksheet 1

Linear Algebra – Refresher

Problem 1. Compute the eigenvalues and eigenvectors of the following matrix \mathbf{A} by hand and using Matlab, Octave or python/numpy. Hand in both your derivation and your code.

$$\mathbf{A} = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$$

Problem 2. Suppose some matrix $\mathbf{B} \in \mathbb{R}^{n \times n}$ has n linearly independent eigenvectors $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n$. Define the matrix \mathbf{U} having these vectors as columns and the diagonal matrix \mathbf{D} with the eigenvalues λ_i in the diagonal. Show that

$$\mathbf{B} = \mathbf{U}\mathbf{D}\mathbf{U}^{-1}$$

Problem 3. For a real symmetric matrix $\mathbf{B} \in \mathbb{R}^{n \times n}$, show that

- \mathbf{B} has real eigenvalues.
- its eigenvectors are orthogonal if the eigenvalues are pairwise distinct.

Problem 4. For a real symmetric matrix $\mathbf{B} \in \mathbb{R}^{n \times n}$ with distinct eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$, show that for $|\cdot|$ (*Determinant*) and $tr(\cdot)$ (*Trace*) the following holds respectively:

$$|\mathbf{B}| = \prod_i \lambda_i$$

$$tr(\mathbf{B}) = \sum_i \lambda_i$$

Problem 5. A *hyperplane* or *affine set* H in \mathbb{R}^n defined by the equation $h(\mathbf{x}) \equiv w_0 + \mathbf{w}^T \mathbf{x} = 0$ (in \mathbb{R}^2 , this is a line). Show that

- for any point $\mathbf{x}_0 \in H$, $\mathbf{w}^T \mathbf{x}_0 = -w_0$.
- if \mathbf{x}_1 and \mathbf{x}_2 lie in H , then $\mathbf{w}^T(\mathbf{x}_1 - \mathbf{x}_2) = 0$.
- $\hat{\mathbf{w}} = \mathbf{w}/\|\mathbf{w}\|$ is the vector normal to the surface of H .
- if $\mathbf{x}_0 \in H$, then the signed distance of any point \mathbf{x} to H is given by

$$\hat{\mathbf{w}}^T(\mathbf{x} - \mathbf{x}_0) = (\mathbf{w}^T \mathbf{x} + w_0)/\|\mathbf{w}\|$$
